Conclusions

The offset-aim target seeker technique is a rapid, effective method for obtaining the launch conditions for fixed time-of-travel interplanetary trajectories. It lends itself well to the employment of simplified methods because it corrects for any biases so introduced. However, accurate, repeatable, and continuous ephemeris, trajectory integration, and Keplerian programs must be available for use with it.

For the examples given, an unperturbed Keplerian orbit is, as expected, a fairly good approximation to the exact interplanetary trajectory that includes the gravitational effects of the solar system planets. Such an orbit provides an excellent first estimate and computational tool for carrying out the offset-aim target seeker method.

Additional applications of this technique are being considered for the following: 1) trajectories between the Earth and the moon; 2) trajectories that satisfy constraints more complex than those just described; 3) multi-leg nonstop trajectories that leave the Earth, pass one or more of the other planets, and return to the Earth; and 4) calculation of interplanetary guidance corrections.

References

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Spreading of Liquid-Surface Jets Supported by Buoyancy Forces

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THE spreading of two-dimensional free jets in laminar and turbulent flows has been studied by H. Schlichting, W. Bickley, W. Tollmien, and H. Goertler and the results are summarized in Ref. 1. In this note, the free jets results will be extended to liquid-surface jets that are lighter than the bulk of the liquid due to a temperature difference.

The static pressure at a point inside the liquid, relative to that in the gas space above, may be written as

$$p(x,y) = g \int_0^y \rho(x,y)dy \tag{1}$$

where ρ is the liquid density and g is the local gravitational acceleration (in the y direction). Differentiating Eq. (1) with respect to x and introducing the volumetric expansion coefficient β, it can be shown for small temperature differences that

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = g\,\beta\,\frac{\partial}{\partial x}\int_0^y (T - T_\infty)dy \tag{2}$$

where T is the local temperature and T_∞ is the temperature outside the mixing layer.

Introducing the following dimensionless variables $U = u/u_1$, $V = v/u_1$, $\Theta = (T - T_\infty)/(T_1 - T_\infty)$, $X = u_1x/\nu$, $Y = u_1y/\nu$, $Pr = C_p\rho\nu/k$, and $a = g\beta(T_1 - T_\infty)\nu/u_1^3$, the boundary-layer equation for laminar flows can be written as

$$(\partial U/\partial X) + (\partial V/\partial Y) = 0 \tag{3}$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = a\frac{\partial}{\partial X}\int_0^Y \Theta dY + \frac{\partial^2 U}{\partial Y_2} \tag{4}$$

$$U\frac{\partial \Theta}{\partial X} + V\frac{\partial \Theta}{\partial Y} = \frac{1}{Pr}\frac{\partial^2\Theta}{\partial Y^2} \tag{5}$$

where u_1 and T_1 are the initial velocity and temperature of the liquid surface jet; u and v are the horizontal and vertical components of velocity; and ν, k, C_p, Pr are kinematic viscosity, thermal conductivity, specific heat at constant pressure, and Prandtl number, respectively. The boundary conditions are

$$\text{At } Y = 0: V = 0 \qquad \frac{\partial U}{\partial Y} = 0 \qquad \frac{\partial \Theta}{\partial Y} = 0$$

$$\text{As } Y \to \infty: U = \Theta \to 0$$

It is assumed that the free surface is adiabatic and that the drag by the gas above the free surface is zero. The momentum and energy integrals may be obtained by integration of Eqs. (4) and (5), yielding

$$\int_0^\infty U^2 dY - a\int_0^\infty \int_0^Y \Theta dY dY = \frac{u_1h}{\nu} - \frac{a}{2}\left(\frac{u_1h}{\nu}\right)^2 \equiv M \tag{6}$$

$$\int_0^\infty U\Theta dY = \frac{u_1h}{\nu} \equiv E \tag{7}$$

where h is the initial width of the surface jet. Equations (6) and (7) also define M and E.

A similar solution for the system of Eqs. (3–7) with the boundary conditions given does not exist. However, a similar solution for laminar free jets ($a = 0$) exists with $\psi \sim X^{1/3}F(\eta)$, $U \sim X^{-1/3}F'(\eta)$, and $\eta = Y/X^{2/3}$, where ψ is the stream function and $F(\eta)$ is a function of the similarity variable η. From Eq. (7), $\Theta \sim X^{-1/3}L(\eta)$, where $L(\eta)$ is a function of η, in order to satisfy the requirement that the energy integral is independent of X. For $a \neq 0$, one may consider the following series expansions:

$$F(\eta, aX) = \frac{\psi}{X^{1/3}} = F_0(\eta) + aXF_1(\eta) + (aX)^2F_2(\eta) + \ldots \tag{8}$$

$$L(\eta, aX) = \Theta X^{1/3} = L_0(\eta) + aXL_1(\eta) + (aX)^2L_2(\eta) + \ldots \tag{9}$$

With $U = (\partial\psi/\partial Y)_X$, $-V = (\partial\psi/\partial X)_Y$, substituting into Eqs. (3–7), collecting terms, and equating the coefficients of the powers of (aX) to zero, yields

$$3F_0''' + (F_0')^2 + F_0''F_0 = 0 \tag{10}$$

$$3F''' + (3n-2)\int_0^\eta L_{n-1}d\eta - 2\eta L_{n-1} = \sum_{k=0}^n [(3k-1) \times F_k'F_{n-k}' - (3k+1)F_kF''_{n-k}] \qquad n \geqq 1 \tag{11}$$

$$\frac{3}{Pr}L_0'' + F_0'L_0 + F_0L_0' = 0 \tag{12}$$

$$\frac{3}{Pr}L_n'' = \sum_{k=0}^n [(3k-1)L_kF'_{n-k} - (3k+1)F_kL'_{n-k}] \qquad n \geqq 1 \tag{13}$$

$$\int_0^\infty (F_0')^2 d\eta = M = \frac{u_1h}{\nu} - \frac{a}{2}\left(\frac{u_1h}{\nu}\right)^2 \tag{14}$$

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$$\sum_{k=0}^{n} \int_0^\infty F_k' F'_{n-k} d\eta = \int_0^\infty \int_0^\eta L_{n-1} d\eta d\eta \qquad n \geqq 1 \quad (15)$$

$$\int_0^\infty F_0' L_0 \, d\eta = E = \frac{u_1 h}{\nu} \quad (16)$$

$$\sum_{k=0}^{n} \int_0^\infty F_k' L_{n-k} \, d\eta = 0 \qquad n \geqq 1 \quad (17)$$

The boundary conditions are

$$\begin{aligned} &\text{At } \eta = 0: \quad F_n(0) = F_n''(0) = L_n'(0) = 0 \\ &\text{As } \eta \to \infty: \quad F_n'(\infty) = L_n(\infty) \to 0 \end{aligned} \qquad n \geqq 0 \quad (18)$$

The solution F_0 is similar to that for a free jet,

$$F_0 = 3^{2/3} M^{1/3} \tanh\left[\frac{1}{2(3)^{1/3}} M^{1/3} \eta\right] \quad (19)$$

Equation (12) is linear in L_0 and can be integrated to yield

$$L_0 = L_0(0) \exp\left[-\frac{Pr}{3} \int_0^\eta F_0 \, d\eta\right] \quad (20)$$

The surface value $L_0(0)$ may be determined by substituting Eqs. (19) and (20) into Eq. (16). After integrating and simplifying,

$$L_0 = \frac{E}{3^{2/3} M^{1/3} I(Pr)} \left\{\cosh\left[\frac{1}{2(3)^{1/3}} M^{1/3} \eta\right]\right\}^{-2Pr} \quad (21)$$

where

$$I(Pr) = \int_0^\infty (\cosh \xi)^{-2(1+Pr)} \, d\xi \quad (22)$$

Equation (11) is linear in F_n and is uncoupled to Eq. (13); therefore, it can be integrated independently. Equation (15) provides a relationship to determine $F_n'(0)$. Equation (13) is linear in L_n and can be integrated after F_n is solved from Eqs. (11) and (15). Finally, Eq. (17) determines $L_n(0)$. The integration may be carried by analog computers or by numerical methods. The recursion formulas enable the computation of the successive terms in the series of Eqs. (8) and (9).

The forementioned analysis may be extended to the corresponding case in turbulent flow. For this case, the kinematic viscosity ν and the thermal conductivity k are replaced by the eddy kinematic viscosity ϵ and eddy conductivity κ, respectively. Assuming that the eddy thermal diffusivity and the eddy kinematic viscosity are equal and using Prandtl's second hypothesis, one may write

$$\frac{\kappa}{\rho C_p} = \epsilon = \epsilon^* \frac{b u_0}{b^* u_0^*} = \epsilon^* \left(\frac{x}{x^*}\right)^{1/2} = \epsilon^* \left(\frac{X}{X^*}\right)^{1/2} \quad (23)$$

where b is the width of the mixing zone, u_0 is the liquid surface velocity, * is the value at a reference distance x^*, $X = u_1 x/\epsilon^*$, and $X^* = u_1 x^*/\epsilon^*$. According to Ref. (1), $b \sim x$, $u_0 \sim x^{-1/2}$, and hence $bu_0 \sim x^{1/2}$ for a turbulent free jet. Introducing the same dimensionless variables used previously for the laminar case except with ν replaced by ϵ^* and $1/Pr$ replaced by $\kappa/C_p\rho\epsilon^*$, one obtains the following momentum and energy equations, respectively:

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = a \frac{\partial}{\partial X} \int_0^Y \Theta dY + \left(\frac{X}{X^*}\right)^{1/2} \frac{\partial^2 U}{\partial Y^2} \quad (24)$$

$$U \frac{\partial \Theta}{\partial X} + V \frac{\partial \Theta}{\partial Y} = \left(\frac{X}{X^*}\right)^{1/2} \frac{\partial^2 \Theta}{\partial Y^2} \quad (25)$$

The momentum and energy integral equations are the same as for the laminar case.

Again, a similar solution does not exist for the turbulent case. However, a similar solution for turbulent free jets ($a = 0$) exists with $\psi \sim X^{1/2}F(\eta)$, $U \sim X^{-1/2}F'(\eta)$, and $\eta = Y/X$. From Eq. (7), $\Theta \sim X^{-1/2}L(\eta)$ in order to satisfy the requirement that the energy integral is independent of X. For $a \neq 0$, one may consider the following series expansions:

$$F(\eta, aX^c) = \frac{\psi}{X^{1/3}} = F_0(\eta) + (aX^c)F_1(\eta) + (aX^c)^2 F_2(\eta) + \ldots \quad (26)$$

$$L(\eta, aX^c) = \Theta X^{1/2} = L_0(\eta) + (aX^c)L_1(\eta) + (aX^c)^2 L_2(\eta) + \ldots \quad (27)$$

where c is a constant to be determined. Following the same procedure for the laminar case, one obtains exactly Eqs. (10–18), if $c = \frac{3}{2}$, $X^* = \frac{4}{9}$, and $Pr = 1$. Since X^* is an arbitrary reference quantity, it is permissible to use the value $\frac{4}{9}$. Therefore, the solution for the turbulent case may be obtained readily once the solution for the laminar case with $Pr = 1$ is obtained.

Reference

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General Asymptotic Suction Solution of the Laminar Compressible Boundary Layer with Heat Transfer

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For very large suction velocities at the wall, the asymptotic solution of the steady compressible two-dimensional laminar boundary layer over a surface of negligible curvature is derived under the following general conditions: an arbitrary prescribed axial pressure gradient, variable suction velocity, an arbitrary prescribed variable wall temperature, variable density, an arbitrary Mach number, a constant but arbitrary Prandtl number, and constant specific heats but variable coefficients of viscosity and, hence, of heat conductivity. It is shown that the dimensionless asymptotic velocity and temperature profiles remain the same regardless of the pressure gradient and of the variability of both the suction velocity and the wall temperature. A Reynolds analogy for this solution is demonstrated. Finally, comparison of the asymptotic solution is made with recent numerical similarity solutions.

Nomenclature

c_p	= specific heat at constant pressure
f_w	= $[2/(m+1)](v_w/u_1)R_x^{1/2}$ (Ref. 8)
k	= coefficient of heat conductivity
L	= characteristic streamwise length
M	= $u/[(\gamma-1)c_pT]^{1/2}$ = Mach number
Pr	= $\mu c_p/k$ = Prandtl number
R_L	= $\rho_1 u_1 L/\mu_1$ = Reynolds number
R_x	= $\rho_1 u_1 x/\mu_1$ = Reynolds number
T	= absolute temperature

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